ELECTRICAL

ENGINEERING

N69-10382	
(ACCESSION NUMBER)	(THRU)
53	1
(PAGES)	(CODE)
CR-98073	31
(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)

FACILITY FORM 602

AUBURN RESEARCH FOUNDATION

AUBURN UNIVERSITY

AUBURN, ALABAMA

DIGITAL COMPENSATION OF THE THRUST VECTOR CONTROL SYSTEM

PREPARED BY

SAMPLED-DATA CONTROL SYSTEMS GROUP

AUBURN UNIVERSITY

C. L. Phillips, Project Leader

SIXTH TECHNICAL REPORT

28 May 1966 to 28 September 1966

CONTRACT NAS8-11274
GEORGE C. MARSHALL SPACE FLIGHT CENTER
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
HUNTSVILLE, ALABAMA

APPROVED BY:

C H Holmes

C. H. Holmes
Head Professor
Electrical Engineering

SUBMITTED BY:

C. L. Phillips

C. L. Phillips
Professor of Electrical Engineering

TABLE OF CONTENTS

LIST OF FIGURES........ ii

LIST OF SYMBOLS........ iii

FOREWORD........ vi

SUMMARY........ vii

PERSONNEL........ viii

I. INTRODUCTION........ 1

II. NUMERICAL INACCURACY IN COMPUTATION OF NYQUIST DIAGRAM DATA BY THE Z-TRANSFORM METHOD........ 3

III. AN UPPER BOUND ON QUANTIZATION ERROR IN DIGITAL SYSTEMS BY THE DIRECT METHOD OF LIAPUNOV........ 10

IV. DERIVATION OF THE TRANSFER FUNCTION, $\phi_D(s)/\beta_c(s)$, FROM THE SATURN V EQUATIONS OF MOTION........ 29

V. CONCLUSIONS........ 33

REFERENCES........ 35

APPENDIX........ 36

LIST OF FIGURES

1. Basic Digital Control System.................................. 12

2. Quantizer Input-Output Relationship........................ 14

3. Signal Flow Graph for Basic Digital Control System with One Quantizer.. 14

4. Signal Flow Graph Representation of Saturn V S-IC Equations of Motion with Slosh Neglected................ 31

LIST OF SYMBOLS

C_1 = Aerodynamic moment coefficient

C_2 = Control moment coefficient

F = Total thrust of vehicle

I_E = Engine moment of inertia about gimbal point

I_{xx} = Pitch plane moment of inertia about vehicle c.g.

m = Total mass of vehicle

m_i = Generalized mass

N' = Aerodynamic force

R' = Thrust of control engines

S_E = First moment of swivel about gimbal point for one engine

$\overline{V}$ = Velocity of vehicle

X = Drag force

k_3 = (F - X)/m

k_4 = R'/m

k_7 = N'/m

X_D = Station of displacement gyro

X_β = Station of engine gimbal

$Y_i(X_\beta)$ = Normalized displacement at station

$Y'_i(X_\beta)$ = Normalized slope at station X_β due to i^{th} mode

$Y'_i(X_D)$ = Normalized slope at station X_D

Z = Displacement normal to reference

β = Total engine deflection

β_c = Input to controller

η_i = Generalized displacement of the i^{th} mode

ϕ = Attitide angle

ϕ_D = Platform angle

ζ_i = Bending mode damping of i^{th} mode

ω_i = Frequency of the i^{th} bending mode

u(kT) = Input to quantizer at k^{th} sampling instant.

ω(kT) = Output of quantizer at k^{th} sampling instant.

q(kT) = Round off error due to quantization at k^{th} sampling instant.

Δ = Quantizer granularity.

Δe(kT) = System error at k^{th} sampling instant due to quantization.

ΔE(z) = z-transform of the sequence, Δe(kT).

q(z) = z-transform of the sequence, q(kT).

$\underline{x}$(kT) = State vector for quantized system.

A,G,H = Matrices used in the state representation of the quantized system.

λ = Symbol representing a matrix eigenvalue.

V($\underline{x}$(kT)) = The Liapunov function evaluated at system state, $\underline{x}$(kT).

∇V($\underline{x}$(kT)) = The first difference of the Liapunov function at $\underline{x}$(kT).

B,C = Positive definite matrices used in the construction of the Liapunov function.

E^n = The state space where a point is defined by the sequence of components of $\underline{x}$(kT).

ρ, $\hat{\rho}$ = Scalar quantities representing the radius of a shperical ball in E^n.

ϵ = Symbol meaning " is a member of."

$\forall$ = Symbol meaning "for all."

$\subset$ = Symbol meaning " is a subset of."

M = A set of points in E^n.

M^c = If $\underline{x} \in E^n$ and x is not a member of M, then $x \in M^c$, the complement of M.

M_r, M_r^c = A set in E^n and its complement.

$|| \underline{x}(kT) ||$ = The norm of the state vector, $\underline{x}(kT)$.

$\underline{\epsilon}(kT)$ = The quantization error state vector.

$| \Delta e_t |$ = The maximum system error due to r quantizers.

FOREWORD

This report is a technical summary of the progress made since May 28, 1966, by the Electrical Engineering Department, Auburn, University, toward fulfillment of Contract No. NAS8-11274 granted to Auburn Research Foundation, Auburn, Alabama. The contract was awarded May 28, 1964 by the George C. Marshall Space Flight Center, National Aeronautics and Space Administration, Huntsville, Alabama.

SUMMARY

A development is given which provides a plausible explanation for the high numerical accuracy often required when z-domain expressions are used to obtain frequency response data for sampled systems. Further, a method is described with which an estimate of the required computational accuracy can be made for a class of sampled systems.

When a digital element is introduced into a system, it is necessary to evaluate the effect on system performance of the resulting representation of some of the system variables by a finite word length. The relative merits of several analytical methods for determining quantization errors are discussed and one of these methods, which is based on Liapunov's Direct Method, is described in detail.

A digital computer program is developed to determine the transfer function from the actuator command signal to the attitude error angle, the roots of the characteristic equation, and the sampled frequency response for the Saturn V.

LIST OF PERSONNEL

The following named staff members of Auburn University have actively participated on this project:

C. H. Holmes- Head Professor of Electrical Engineering

G. T. Nichols- Associate Professor of Electrical Engineering

C. L. Phillips- Professor of Electrical Engineering

R. K. Cavin III- Graduate Assistant in Electrical Engineering

D. L. Chenoweth- Graduate Assistant in Electrical Engineering

J. C. Johnson- Graduate Assistant in Electrical Engineering

DIGITAL COMPENSATION OF THE THRUST VECTOR CONTROL SYSTEM

I. INTRODUCTION

A problem of recurring interest in the analysis of sampled data systems is that erratic data is often obtained when the frequency response is computed in the z-domain. Frequently, as many as 24 significant figures must be used in order to generate data from z-domain expressions. In Chapter II, it is shown that as the poles of a z-transform expression are placed in closer proximity to one another, the number of significant digits required for computation of valid results increases. Moreover, criteria are developed from which the required number of digits can be calculated based on the pole locations for a particular class of systems.

The effects of quantization on the response of discrete systems is considered in Chapter III. Quantization is a result of the digital computer characteristic that all words must have a finite length; i.e., each variable is represented by a finite number of digits. It is shown that these round off nonlinearities will not destabilize the system in the bounded input-bounded output sense. However, system errors and/or limit cycle behaviour may result. A recently developed method for the determination of bounds on dynamic system errors caused by quantization is discussed. This procedure is based on the Direct Method of Liapunov.

The problem of developing a transfer function from the control engine command signal to the angular attitude error is considered in Chapter IV. Signal flow graph techniques are used to develop the required transfer function from the planar Saturn V equations of motion with vehicle slosh dynamics neglected. A computer program is presented which performs the following functions:

(1) Computes the transfer function coefficients,

(2) Factors the system characteristic equations, and,

(3) Computes the sampled frequency response of the transfer function.

II. NUMERICAL INACCURACY IN COMPUTATION OF NYQUIST DIAGRAM DATA BY THE Z-TRANSFORM METHOD

1. Introduction

A problem of numerical inaccuracy often arises in the computation of Nyquist diagram data, using a digital computer, for sampled-data systems by the z-transform method. A cause of this problem is investigated below, and methods for averting the resulting numerical inaccuracies are given.

2. Case I - Two Poles Almost Coincident

Suppose that the continuous-data open-loop transfer function of a sampled-data system is given by

$$G(s) = \frac{k_1}{s+a_1} + \frac{k_2}{s+a_2} + \frac{k_3}{s+a_3} + \ldots \tag{2-1}$$

The Nyquist-diagram data is obtained by evaluating G(z), the z-transform of G(s), for values of z on the unit circle. A problem of numerical inaccuracy arises when two or more poles of G(s) are nearly coincident.

Suppose that, in (2-1),

$$a_2 = a_1 + \Delta , \qquad |\Delta| > 0, \tag{2-2}$$

where $|\Delta|$ is small. Suppose further that there are intervals of the unit circle in the z-plane for which all other terms of the partial-fraction expansion of G(z) are small compared to the first two terms, i.e., for these intervals of the unit circle,

$$G(z) \cong \mathcal{Z}\left[\frac{k_1}{s+a_1} + \frac{k_2}{s+a_2}\right] = \mathcal{Z}\left[\frac{b_1 s+b_0}{(s+a_1)(s+a_2)}\right]. \tag{2-3}$$

Then, expressing k_1 and k_2 in terms of b_1, b_0, a_1, and a_2,

$$\begin{aligned} G(z) &= \mathcal{Z}\left[\frac{-b_1a_1 + b_0}{(a_2-a_1)(s+a_1)} + \frac{-b_1a_2 + b_0}{(a_1-a_2)(s+a_2)}\right] \\ &= \frac{\left[\frac{b_0-b_1a_1}{\Delta}\right] z}{z-\epsilon^{-a_1T}} - \frac{\left[\frac{b_0-b_1a_2}{\Delta}\right] z}{z-\epsilon^{-a_2T}} \quad . \end{aligned} \tag{2-4}$$

It is noted that as $a_1 \rightarrow a_2$, $\Delta \rightarrow 0$,and the numerator coefficients in (2-4) become unbounded. However, as $\Delta \rightarrow 0$,

$$\begin{aligned} \lim_{\Delta \rightarrow 0} G(z) &= \mathcal{Z}\left[\frac{b_1 s+b_0}{(s+a_1)^2}\right] = b_1\left[\frac{1-\epsilon^{-a_1T}(1 + a_1T)z^{-1}}{(1-\epsilon^{-a_1T}\, z^{-1})^2}\right] \\ &+ b_0 \left[\frac{T\epsilon^{-a_1T}z^{-1}}{(1-\epsilon^{-a_1T}z^{-1}}\right] , \end{aligned} \tag{2-5}$$

and G(z) is bounded for values of z on the Nyquist path, except, of course, for the case that the pole location is on the path. Thus, for Δ small, the value of G(z) is obtained by subtracting two large numbers which are almost equal. If the difference of the two large numbers is affected appreciably by round off error, the data obtained for the Nyquist diagram will fluctuate in some random manner and will be incorrect.

It is seen from the above discussion that the inaccuracies are greater where the two parts of (2-4) are most nearly equal in magnitude. If the pole location, ϵ^{-a_1T}, is in the neighborhood of $z = 1$, which will be the case for a system which is low-pass with respect to the sampling frequency, then the two parts of (2-4) are most nearly equal in magnitude at $z = -1$. The difference in the two values of (2-4) is

$$\left[\frac{b_0 - b_1a_1}{\Delta} - \frac{b_0 - b_1a_2}{\Delta}\right] \frac{1}{1 + \epsilon^{-a_1T}} = \frac{b_1}{1 + \epsilon^{-a_1T}} \tag{2-6}$$

for $b_1 \neq 0$. Then the accuracy of the calculation can be determined by comparing the magnitude of $(b_0-b_1a_1)/\Delta$ to b_1. The difference between the values of $G(z)$ in (2-4) occurs in the n^{th} significant figure of the computation, where n is given by

$$\left|\frac{b_0 - b_1a_1}{\Delta b_1}\right| \cong 10^n \quad , \tag{2-7}$$

and where (2-7) is satisfied only to the order of magnitude. The value of n computed from (2-7) is an indication of the minimum number of significant figures which must be used in order to obtain accurate frequency response data.

If b_1 is zero, then, for $z = -1$, (2-4) becomes

$$G(-1) = \frac{\frac{b_0}{\Delta}(-1)}{-1-\epsilon^{-a_1T}} - \frac{\frac{b_0}{\Delta}(-1)}{-1-\epsilon^{-a_2T}} = \frac{\frac{b_0}{\Delta}(\epsilon^{-a_2T}-\epsilon^{-a_1T})}{(1+\epsilon^{-a_1T})(1-\epsilon^{-a_2T})} . \tag{2-8}$$

Numerical inaccuracies occur when the magnitude of one of the terms of (2-8) is large compared to the total value of (2-8). The difference between the values of the terms of (2-9) occurs in the n^{th} significant figure of the computation, where n is given by

$$\left| \frac{1-\epsilon^{-a_1T}}{\epsilon^{-a_2T}-\epsilon^{-a_1T}} \right| \cong 10^n , \tag{2-9}$$

and where (2-9) is satisfied only to the order of magnitude.

3. Case II - Three Poles Almost Coincident

For Case II, it is assumed that over certain intervals of the z-plane unit circle G(z) of (2-3) can be approximated by

$$G(z) \cong \mathcal{Z}\left[\frac{b_2s^2+b_1s+b_0}{s(s+a_1)(s+a_2)}\right] , \tag{2-10}$$

where a_1 and a_2 are small. The assumption is made that the poles are located in the neighborhood of the origin since, for the sampled-data thrust vector control system, two poles of G(s) occur near the origin and the zero-order hold places a pole at the origin. If a higher ordered data hold is used, then higher ordered poles occur at the origin, and the problem is further complicated. This was the case of the example in the Third Technical Report.* Now, from (2-10),

$$G(s) \cong \frac{b_2 s^2 + b_1 s + b_0}{s(s+a_1)(s+a_2)} = \frac{k_1}{s} + \frac{k_2}{s+a_1} + \frac{k_3}{s+a_2}, \tag{2-11}$$

where

$$\begin{aligned} k_1 &= \frac{b_0}{a_1 a_2} \\ k_2 &= \frac{b_2 a_1^2 - b_1 a_1 + b_0}{a_1(a_1 - a_2)} \\ k_3 &= \frac{b_2 a_2^2 - b_1 a_2 + b_0}{a_2(a_1 - a_2)} . \end{aligned} \tag{2-12}$$

Then

$$G(z) \cong \frac{k_1 z}{z-1} + \frac{k_2 z}{z-\epsilon^{-a_1 T}} + \frac{k_3 z}{z-\epsilon^{-a_2 T}} . \tag{2-13}$$

As $z \rightarrow -1$ on the Nyquist path, the denominators of the terms of (2-13) approach each other in value. Or

$$\lim_{z \rightarrow -1} G(z) \cong \frac{z}{z-1}(k_1 + k_2 + k_3) = \frac{1}{2}(k_1 + k_2 + k_3) \tag{2-14}$$

*Phillips, C. L., et. al., "Digital Compensation of the Thrust Vector Control System," Third Technical Report, Contract NAS8-11274, Auburn Research Foundation, Auburn, Ala., pp. 47-55; September 28, 1965.

Now, from (2-12)

$$k_1 + k_2 + k_3 = b_2 \ . \tag{2-15}$$

Numerical inaccuracies occur, then, if b_2 is small in magnitude compared to either k_1, k_2, or k_3. For example, if $|a_1| << 1$ and $|a_2| << 1$, then

$$|k_1| = \left|\frac{b_0}{a_1 a_2}\right| >> 1 \quad , \tag{2-16}$$

and $|k_1|$ could be several orders of magnitude larger than b_2. The difference between the values of the terms of (2-14) occur in the n^{th} significant figure of the computation, where n is the largest value given by the relationships

$$\begin{aligned} \left|\frac{k_1}{b_2}\right| &\cong 10^n \\ \left|\frac{k_2}{b_2}\right| &\cong 10^n \\ \left|\frac{k_3}{b_2}\right| &\cong 10^n \quad , \end{aligned} \tag{2-17}$$

and where (2-17) are satisfied only to the order of magnitude. If b_2 is zero, then expressions equivalent to (2-17) can be obtained is terms of b_1 and b_0.

4. Methods to Avert Inaccuracies

There are several methods by which the inaccuracies discussed above may be averted. These methods are given below.

1. If numerical inaccuracies occur when the computations are carried out in single precision (eight significant figures), performing the calculations in either double precision or triple precision will usually give the necessary accuracy.
2. For Case I, it is seen from (2-7) that the two poles must be very close together before inaccuracy problems will occur. For this case, the poles may be assumed to be exactly coincident with no noticable error in the Nyquist diagram. Then no inaccuracy problem will arise. For Case II, inaccuracies can occur with either a_1 or a_2 relatively large. For this case, errors will occur if the poles are assumed to be coincident.
3. The z-transform of the partial fractions causing the inaccuracy problems, as in (13), may be recombined into a higher-ordered function of z using double or triple precision, which ever is necessary. Then the Nyquist diagram data may be obtained using single precision.
4. The total z-transform of G(s) may be recombined into a single rational fraction in z using double or triple precision. Then the Nyquist diagram data may be obtained using single precision.

III. AN UPPER BOUND ON QUANTIZATION ERROR IN DIGITAL SYSTEMS BY THE DIRECT METHOD OF LIAPUNOV

1. INTRODUCTION

This chapter is based almost entirely on the work of G. W. Johnson as given in [1,2]. The purpose of the following discussion is to develop more explicitly some of the relations used in [1,2] and to define some of the terminology used by Johnson in the context of engineering usage.

Johnson's method is applicable to stable discrete systems which are linear with the exception of quantizing non-linearities. His results are particularly attractive in that a realistic error bound is established in closed form for both transient and steady state errors.

In general, the solutions to the problem of quantization error estimation have fallen into two categories. Widrow [3,4] considered the problem from a statistical viewpoint and he established that, if certain criteria are satisfied, an r m s error estimate can be obtained. This method does not, however, provide a worst case error bound and is not applicable if the input signal to the quantizer does not have a dynamic range covering several quantizer steps.

The other category of error estimation, and the one to which Johnson's method belongs, is that of establishing an upper bound on system errors due to quantization. Bertram [5] developed a technique with which a rather pessimistic error bound may be obtained. The method may be used to obtain both dynamic and steady state bounds for the error but, unless the system response is overdamped, a separate computation is required for each sampling instant. Slaughter [6] advanced a technique which also falls into this second category. This technique is very much like the one given by Bertram, differing only in that the final value theorem is applied to obtain an estimate of steady state system error due to quantization. The resulting steady state upper bound is somewhat less pessimistic than Bertram's steady state error bound but this bound is, again, not necessarily an upper bound on transient errors unless the system response is overdamped.

2. Mathematical Formulation

The system given by Figure 1 will be used as a basis for the theoretical discussions which follow. Note that only one quantizer, at the output of the digital element, is shown in this figure. Although the relations to be developed are extendable to multiple quantizer systems, (Sect. 4), the developmental work is somewhat less cluttered if only one quantizer is considered. Further, it is quite often the case that a single quantizer, usually the output converter, is sufficiently gross relative to other system quantizers so that only its effects need to be considered.

After Johnson, let $u(kT)$ represent the input to the quantizer, $w(kT)$ the output and $q(kT)$ the round off error due to quantization. Thus

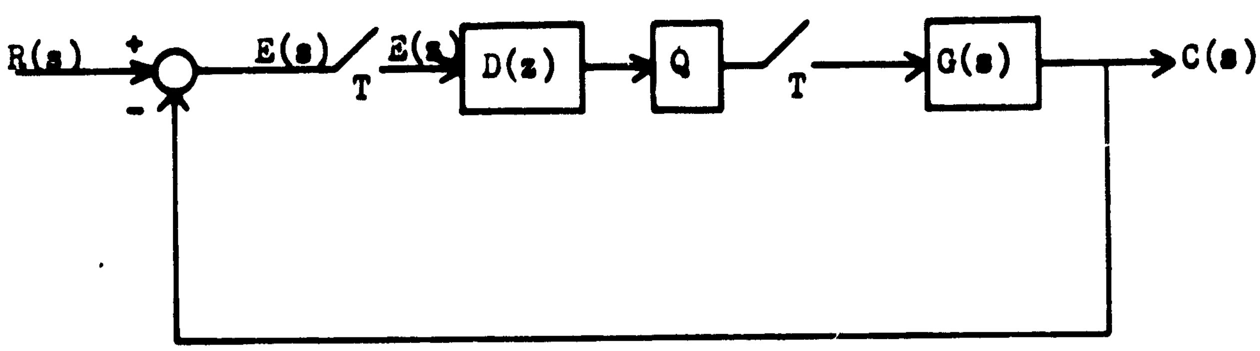

Figure 1.-Basic Digital Control System

$$q(kT) = u(kT) - w(kT) . \tag{3-1}$$

Referring now to Figure 2,

$$w(kT) = (i) \cdot \Delta , \tag{3-2}$$

where i is an integer with the property that

$$(i - \frac{1}{2})\Delta \leq u(kT) < (i + \frac{1}{2})\Delta . \tag{3-3}$$

Thus, the quantizer shown in Figure 1 can be replaced by a unity gain element whose output is summed with the quantizer error, q(kT). Note that in order to obtain a precise description of system response, it is necessary to determine the quantizer error sequence, q(kT). In general this is difficult to obtain in closed form since the members of this sequence are complex functions of the past system states and inputs. Nevertheless, the system can be represented, conceptually at least, as shown by the signal flow diagram of Figure 3. In this figure, $\Delta E(z)$ is the z-transform of the system error which results from the q(kT) sequence. Further; if q(z) is the z-transform of q(kT),

$$\Delta E(z) = \frac{q(z)\, G(z)}{1 + G(z)\, D(z)} , \tag{3-4}$$

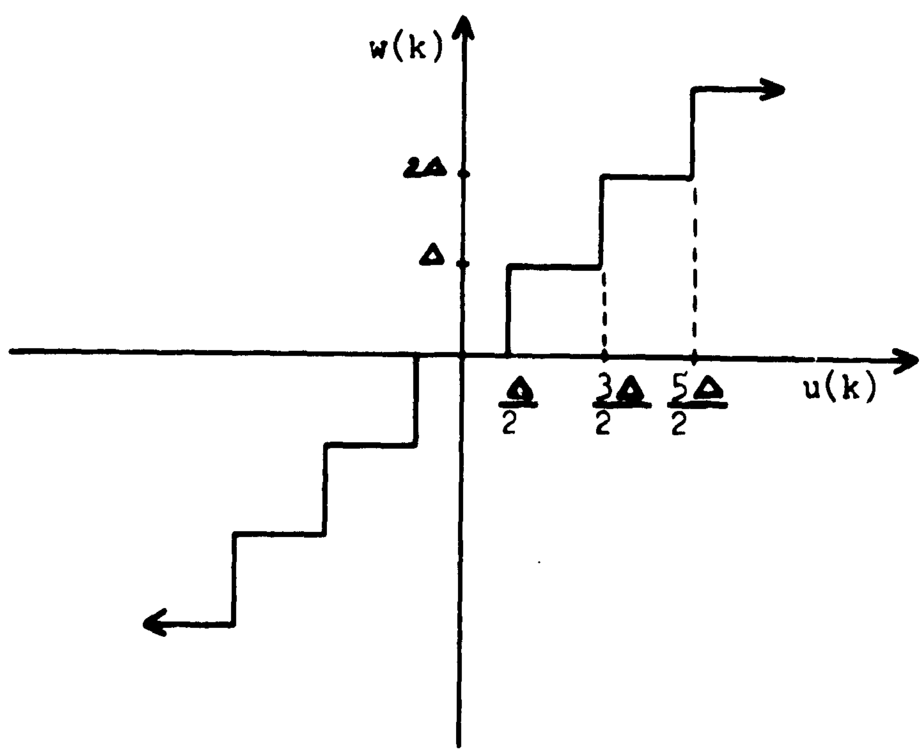

Figure 2.-Quantizer Input-Output Relationship

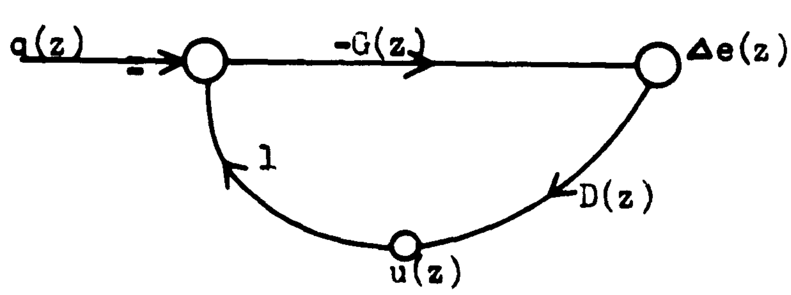

Figure 3.-Signal Flow Graph for Basic Digital Control System with One Quantizer

which in general assumes the following form

$$\triangle E(z) = \frac{(h_m z^m + . . . + h_0) \quad q(z)}{z^n + a_{n-1} z^{n-1} + \ldots + a_0} , \tag{3-5}$$

where $n \geq m$ for physical realizability. Equation (3-5) may be represented by a set of linear difference equations. One technique for accomplishing this transformation is the following. Let

$$M(z) = \frac{q(z)}{z^n + a_{n-1} z^{n-1} + . . . + a_0} = \frac{\triangle E(z)}{h_m z^m + . . . + h_0} . \tag{3-6}$$

Thus,

$$m(k+n) + a_{n-1} m(k+n-1) + . . . + a_0 m(k) = q(k), \tag{3-7}$$

and

$$h_m m(k+m) + . . . + h_0 m(k) = \triangle E(k) . \tag{3-8}$$

Let

$$\begin{aligned} x_1(k) &= m(k) \\ x_2(k) &= m(k+1) \\ &\vdots \end{aligned}$$

$$x_{n-1}(k) = m(k+n-2)$$
$$x_n(k) = m(k+n-1) \quad . \tag{3-9}$$

Combining (3-7) and (3-9), the following state vector difference equations result.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \cdot \\ \cdot \\ \cdot \\ x_n(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ - & - & - & & - & - \\ 0 & 0 & 0 & & 0 & 1 \\ -a_0 & -a_1 & -a_2 & - - - & -a_{n-2} & -a_{n-1} \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ \cdot \\ \cdot \\ \cdot \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} q(k) \quad . \tag{3-10}$$

$\Delta\, e(k)$ can be determined from (3-8) and (3-9).

$$\Delta\, e(k) = h_0\, x_1(k) + \ldots + h_m\, x_m(k) + 0\, x_{m+1}(k) + \ldots$$
$$+ 0\, x_n(k) \quad . \tag{3-11}$$

It will be convenient to write the equations (3-10) and 3-11) in the more compact vector-matrix notation.

$$\underline{x}(k+1) = A\, \underline{x}\, (k) + G\, q(k) \quad . \tag{3-12}$$

$$\Delta e(k) = H^T \underline{x}(k), \tag{3-13}$$

where $H^T = [h_0, h_1, \ldots h_m, 0_{m+1}, \ldots 0_n]$. (3-14)

Recall that, at the outset, it was stated that only stable linear systems would be considered. This means that the eigenvalues of A lie within the unit circle; that is,

$$|\lambda_i [A]| < 1; \quad i = 1, \ldots, n. \tag{3-15}$$

Now the quantizer error sequence, q(k), is bounded for all time, i.e.,

$$|q(k)| \leq \frac{\Delta}{2} \cdot \qquad k \geq k_o \tag{3-16}$$

This leads to the important conclusion that the output error, $\Delta e(k)$ is also bounded for all time. (That is, bounded inputs to a stable linear system will result in bounded outputs.)

Further, The solution to (3-12) and (3-13) is simply,

$$\Delta e(j) = H^T [A^j \underline{x}(0) + \sum_{k=1}^{j} A^{j-k} G\, q(k-1)] \; . \tag{3-17}$$

Now the term $H^T A^j \underline{x}(0)$ decays exponentially to zero as j increases since A^j is the transition matrix of a stable system. It is evident then that the steady state quantization error is determined by the term $H^T \sum_{k=1}^{j} A^{j-k} G\, q(k-1)$.

3. Application of the Direct Method of Liapunov

The direct method of Liapunov attempts to make statements on the stability of the equilibrium without any knowledge of the solutions of the differential (or difference) equations. The theorems developed by Liapunov are applicable for the investigation of the behaviour of an unforced system in the neighborhood of the equilibrium point. It has been shown, however, that if the linear system (unforced) is asymptotically stable in the sense of Liapunov, then it follows that the forced linear system possesses bounded input-bounded output stability [7]. A similar statement is valid for systems giverned by linear difference equations.

Consider the unforced state equation,

$$\underline{x}(k+1) = A\,\underline{x}(k), \tag{3-18}$$

where the eigenvalues of A lie within the unit circle. Let the Liapunov function, $V(\underline{x}(k))$ be of the quadratic form,

$$V(\underline{x}(k)) = \underline{x}^T(k)\;B\;\underline{x}^T(k), \tag{3-19}$$

where B is a positive definite matrix. The first difference of (3-19) must be negative for all $\underline{x}$ in order for $v(\underline{x}(k))$ to be a Liapunov function for the system of (3-18).

$$\nabla V(\underline{x}(k)) = V(\underline{x}(k)) - V(\underline{x}(k)) = \underline{x}^T(k)\;B\;\underline{x}(k) - \underline{x}^T(k-1)\;B\;\underline{x}(k-1)\;. \tag{3-20}$$

However, from (3-18),

$$\underline{x}^T(k) = \underline{x}^T(k-1)\ A^T. \tag{3-21}$$

Thus (3-20) becomes

$$\begin{aligned} \nabla V(\underline{x}(k)) &= \underline{x}^T(k-1)\ A^T\ B\ A\ \underline{x}(k-1) - \underline{x}^T(k-1)\ B\ \underline{x}(k-1) \\ &= \underline{x}^T(k-1)\ [A^T\ B\ A - B]\ \underline{x}(k-1). \end{aligned} \tag{3-22}$$

The requirement that $\nabla V(\underline{x}(k))$ be negative for all $\underline{x}(k)$ is equivalent to the requirement that

$$[A^T BA - B] = -C \tag{3-23}$$

is negative definite. Therefore, C is positive definite. It has been shown that for any choice of C that is positive definite, there exists a unique solution of (3-23) for the elements of B only in case the eigenvalues of A all lie within the unit circle,* a condition which is satisfied for the system under consideration.

The significant aspect of the approach advanced by Johnson is that the direct method of Liapunov is used to investigate the "uniform boundedness" of the solutions of (3-12). Hahn [8] gives a definition of uniform boundedness relating to systems described by differential equations. The definition is restated here for difference equations.

Definition: The solutions of a difference equation are said to be "uniformly bounded", if there exists, for a given $h > 0$, a constant $r > 0$ depending only on h such that,

$$\| \underline{x}(k, x_o, k_o) \| < r \quad \text{for all } k \geq k_o ,$$

if $\| \underline{x}(k_o) \| < h$.

Definition: The notation $\underline{x}(k,\underline{x}_o,k_o)$ is read; the state of the system at time kT starting at state $\underline{x}_o$ and time k_oT, where k and k_o are integers.

Definition: $\underline{x}(k,\underline{x}_o,k_o) = [x_1(k,x_o,k_o), \ldots, x_n(k,x_o,k_o)]^T$. Then $\| \underline{x}(k,\underline{x}_o,k_o) \| = [x_1^2 + \ldots, + x_n^2]^{1/2}$.

Although $V(\underline{x}(k)) = \underline{x}^T(k)\, B\, \underline{x}(k)$ is a Liapunov function for the unforced system equations, (3-18), it can also be shown to be a Liapunov function in the sense of uniform boundedness of solutions by the following arguments. By utilizing equation (3-12), the first difference, $\nabla V(\underline{x}(k))$ can be determined to be

$$\begin{aligned} \nabla V(\underline{x}(k)) &= V(\underline{x}(k+1)) - V(\underline{x}(k)) \\ &= \underline{x}^T(k+1)\, B\, \underline{x}(k+1) - \underline{x}^T(k)\, B\, \underline{x}(k) \\ &= [\underline{x}^T(k)\, A^T + G^T q(k)]\, B\,]A\, \underline{x}(k) + G\, q(k)] \\ &- \underline{x}^T(k)\, B\, \underline{x}(k). \end{aligned}$$

After considerable manipulation, (3-24) becomes

$$\nabla V(\underline{x}(k)) = -\underline{x}^T(k)\ C\ \underline{x}(k) + G^T\ B\ G\ q^2(k) + 2G^T\ B\ A\ \underline{x}(k)\ q(k)\ . \tag{3-25}$$

It is apparent upon inspection of (3-25) that there exists a region in space, E^n, whose axes are $x_1, \ldots, x_n$, such that $\nabla V(\underline{x}(k))$ is negative. This arises because of the second order effect of the first term on the right hand side of (3-25). Thus a spherical ball may be defined with radius ρ such that if $\|\underline{x}\| > \rho$, $\nabla V(\underline{x}(k)) < 0$. A value of ρ which satisfies this condition may be computed from equation (3-25). In order to facilitate the development of ρ, it is necessary to define several terms of relevance to the derivation.

<u>Definition</u>: The symbol $\| A \|$ denotes the spectral norm or simply the norm of matrix A.[9] Further,

$$\| A \| = \sup_{x \neq 0} \frac{\| A\underline{x} \|}{\| \underline{x} \|} \tag{3-26}$$

In words, $\| [A] \|$ is a number with the property that if $\underline{x} \in E^n$; i.e., is a member of E^n, then the norm of the point $A\underline{x}$ is less than or equal to $\| A \| \cdot \| \underline{x} \|$.

Some of the properties of spectral norms of matrices are given by the following theorem taken intact from reference [9].

<u>Theorem</u>: If A and B are two n x n matrices, then $\| A \| > 0$ unless $A = 0$, the null matrix; if α is a scalar;

$$\| \alpha A \| = | \alpha | \cdot \| A \| \tag{3-27}$$

$$\| A + B \| \leq \| A \| + \| B \| \tag{3-28}$$

$$\| A \cdot B \| \leq \| A \| \cdot \| B \| \quad . \tag{3-29}$$

If a matrix, A, is positive definite, it is symmetric about the diagonal and thereby has several useful properties which arise from its structure.[9]

$$\| A \| = [\max_{i=1,n} \lambda_i (A)] \quad , \tag{3-30}$$

i.e., the norm of A is equal to the positive square root of the maximum eigenvalue of A.

Further, if A is positive definite,[11]

$$[\min \lambda_i(A)] \cdot \| \underline{x} \|^2 \leq \underline{x}^T A \underline{x} \leq [\max_{i=1,n} \lambda_i (A)] \cdot \| \underline{x} \|^2 \, . \tag{3-31}$$

The above properties, which are sufficient for the purposes of this discussion, are necessarily a rather restricted summary of the properties of matrix norms.

Now, returning to the problem of establishing the quantity ρ, the requirement that $\nabla V(\underline{x}(k))$ must remain negative is equivalent to requiring that

$$\underline{x}^T(k)\ C\ \underline{x}(k) \geq G^T\ B\ G\ q^2(k) + 2G^T\ B\ A\ \underline{x}(k)\ q(k)\ . \tag{3-32}$$

Now the right hand side of (3-32) is a maximum when q(k) is its maximum value of $\frac{\Delta}{2}$ and chosen with appropriate sense to make the second term on the right hand side of (3-32) positive. Thus (3-32) becomes

$$\underline{x}^T(k)\ C\ \underline{x}(k) \geq G^T\ B\ G\ \frac{\Delta^2}{4} + \left|\ 2G^T\ B\ A\ \underline{x}(k)\ \frac{\Delta}{2}\ \right|\ . \tag{3-33}$$

Now by equation (3-31) ,

$$\underline{x}^T(k)\ C\ \underline{x}(k) \geq \min_i \lambda_i\ (C) \cdot ||\ \underline{x}\ ||^2\ . \tag{3-34}$$

Thus the satisfaction of inequality (3-33) may be guaranteed by requiring that

$$\min_i\ \lambda_i\ [C] \cdot ||\ \underline{x}\ ||^2 \geq G^T\ B\ G\ \frac{\Delta^2}{4} + |G^T\ B\ A\ \underline{x}\ \Delta|\ . \tag{3-35}$$

A simplified expression which assures inequality (3-34) may be realized, is

$$\min_i \lambda_i\ [C] \cdot ||\ \underline{x}\ ||^2 \geq G^T\ B\ G\ \frac{\Delta^2}{4} + ||\ G^T\ B\ A\ \Delta\ || \cdot ||\ \underline{x}\ ||\ . \tag{3-36}$$

It is apparent that if the equality is used in (3-36), the solution of (3-36) for $||\ \underline{x}\ ||$ will represent the least value of $||\ \underline{x}\ ||$ for which (3-33) is satisfied. This corresponds to the quantity ρ, defined previously.

$$\rho = \left[\frac{\| G^T B A \|}{\min\limits_i \lambda_i [C]} + \left(\frac{\| G^T B A \|^2}{\min\limits_i \lambda_i^2 [C]} + \frac{G^T B G}{\min\limits_i \lambda_i [C]} \right)^{1/2} \right] \cdot \frac{\Delta}{2} \tag{3-37}$$

At this point, perhaps it is worthwhile to summarize and consolidate the results obtained. By application of (3-37), a spherical region may be defined in E^n with the property that $\nabla V(\underline{x}(k))$ is negative definite for all values of state exterior to this region. The next objective is to define the region of uniform boundedness of solutions for (3-12), i.e., the bounded region within which the system state always remains. Obviously, since the system error due to quantization is a linear combination of the state of the system at any time k, if it can be shown that there exists a bounded region within which the system state always remains, then it follows that the error due to quantization is always bounded.

This leads to the following theorem which states the central results of Johnson's papers.

<u>Theorem</u>:

Given: The system of linear difference equations (3-12)

(a) The Liapunov function, (3-9).

(b) The number ρ of (3-37) with the property that if $\| \underline{x} \| > \rho$, $\nabla V(\underline{x}(k)) < 0$.

Let M be the set to which $\underline{x}$ belongs only in case $\| \underline{x} \| \leq \rho$. Let M_r

be the set to which $\underline{x}$ belongs only in case

$$\|\underline{x}\| \le \rho \left[\frac{\lambda \max (B)}{\lambda \min (B)} \right]^{1/2} = \hat{\rho}.^{1} \tag{3-38}$$

Then if at $t = k_oT$, $x(k_oT)$ is a member of M, $x(kT)$ is a member of M_r for $kT \geq k_oT$.

Proof: The proof of this theorem is an almost immediate consequence of a lemma on uniform boundedness given in reference [10]. This lemma is stated and proved below.

Lemma: Let $V(\underline{x})$ be a scalar function with continuous first partial derivatives for all $\underline{x}$. Let M be a closed and bounded set in the state space with the property that if $\underline{x} \epsilon M^c$, $\nabla V(\underline{x}) \le 0$. Let M_r be a closed and bounded set with the property that $M \subset M_r$ such that if $\underline{x}_1 \epsilon M$ and $\underline{x}_2 \epsilon M_r^c$, then $V(\underline{x}_1) < V(\underline{x}_2)$. Then each solution of (3-12) which at some time $k_oT \geq 0$ is in M can never thereafter leave M_r.

$$\underline{x}\,(k+1) = A\,\underline{x}\,(k) + G\,q(k)\ . \tag{3-12}$$

($q(k)$ is bounded $\forall$ k).

Proof: (after Lasalle & Lefschetz)

Suppose that a solution of (3-12) is in M at time k_oT. Assume that at k_AT, $\underline{x}(k_AT) \in M_r^c$, $(k_o < k_A)$.

---

[1] It can be shown that $\hat{\rho} \geq \rho$ by using (3-31).

Then there must exist a time k_1T such that $x(kT) \in M^c$ for $k_1T < kT \leq k_AT$. Note this implies that $x(k_1T) \in M$.

From the problem hypothesis,

$$V(\underline{x}(k_1T)) = V(\underline{x}(k_AT)) . \quad (3\text{-}39)$$

This is a contradiction since $\nabla V(\underline{x}) \leq 0$ for $\underline{x} \in M^c$. Therefore, $\underline{x}(kT)$ cannot leave M_r and the lemma is proved.

Thus, all that is needed for a proof of the theorem is a demonstration that conditions imposed in the theorem satisfy the requirements of the lemma. These requirements are:

(1) $\nabla V(\underline{x}) \leq 0$ for $\underline{x} \in M^c$ - This is immediate from the definition of M.

(2) Show that if $\underline{x}_1 \in M$ and $\underline{x}_2 \in {M_r}^c$, $V(\underline{x}_1) < V(\underline{x}_2)$.

The maximum value that $V(\underline{x})$ can attain if $\underline{x} \in M$ can be computed as follows.

$$V(\underline{x}) = \underline{x}^T B\underline{x} \leq \lambda\max\ (B)\ ||\ \underline{x}\ ||^2 . \quad (3\text{-}40)$$

$$\max_{\underline{x} \in M} ||\ \underline{x}\ || = \rho . \quad (3\text{-}41)$$

Therefore,

$$\max_{\underline{x} \in M} V(\underline{x}) \leq \lambda\max\ (B) \cdot \rho^2 . \quad (3.42)$$

The minimum value that $V(\underline{x})$ can obtain may be determined as follows:

$$\underset{\underline{x} \in M_r^c}{V(\underline{x})} \geq \lambda\text{min } (B) \; || \; \underline{x} \; ||^2 \; . \tag{3-43}$$

$$\underset{\underline{x} \in M_r^c}{\text{Min}} \; || \; \underline{x} \; || > \hat{\rho} \quad . \tag{3-44}$$

$$\underset{\underline{x} \in M_r^c}{\text{Min } V(\underline{x})} > \lambda\text{min } (B) \; \hat{\rho}^2 = \lambda\text{min } (B) \cdot \frac{\lambda\text{max } (B)}{\lambda\text{min } (B)} \cdot \rho^2 \; . \tag{3-45}$$

$$\underset{\underline{x} \in M_r^c}{\text{Min } V(\underline{x})} > \lambda\text{max } (B) \cdot \rho^2 \quad . \tag{3-46}$$

Equations (3-46) and (3-42) may be combined to yield the following inequality.

$$\underset{\underline{x} \in M}{\text{Max } V(\underline{x})} < \underset{\underline{x} \in M_r^c}{\text{Min } V(\underline{x})} \quad . \tag{3-47}$$

The conditions imposed by the lemma are satisfied and the theorem is proved.

Thus $|| \; \underline{x}(k) \; || < \hat{\rho}$ for all k. From equation (3-13), the system error is

$$\Delta \; e(k) = H^T \; \underline{x} \; (k) \leq || \; H^T \; || \cdot || \; x(k) \; || < || \; H^T \; || \cdot \hat{\rho} \; . \tag{3.48}$$

Some Implications of the Above Development

It has been shown that if $\underline{x}\epsilon M$ at k_oT, $\underline{x}\epsilon M_r$ for all $k_o \leq k$. A logical question which arises and one which Johnson does not clearly answer is: In the physical system, is $\underline{x}(k_o)$ always a member of the set M? It would seem that $\underline{x}(k_o)$ would be free to assume any value within the dynamic range of the system variables. A further factor seemingly omitted by Johnson's development is the system input, r(kT). Actually both of these considerations are implicitly involved in (3-12). Bertram [5] describes a set of error state variables which are defined as the difference between the quantized state variables and the unquantized variables. For a system with only one quantizer, the following form results

$$\underline{\epsilon}(k+1) = A\underline{\epsilon}(k) + \underline{d}q. \qquad (3\text{-}49)$$

Note that (3-49) is analogous to (3-12). Precisely the same solution for $\Delta e(k)$ will be obtained by using either (3-12) or (3-49). $\Delta e(k)$ is, in fact, the difference in system error e(k) for the quantized and unquantized systems. Further, it is easily shown by using Bertram's arguments that the term r(kT) does not appear in (3-49). Also, it is reasonable to assert that the unquantized and quantized states are initially coincident. It then follows that $\underline{\epsilon}(k_oT) = 0$ (0 is the Null matrix). Therefore $\underline{x}(k_oT) \in M$.

4. The Case for Multiple Quantizers

Johnson states without proof that the method can be extended to the analysis of multi-quantizer systems by simply invoking the triangular rule. This will be shown below. The upper bound on system error due to the j^{th} quantizer is

$$|\Delta e_j(k)| \leq || H_j || \hat{\rho} \qquad \text{for all } k \geq k_o \quad . \tag{3-50}$$

[$\hat{\rho}$ is independent of the choice of quantizer, unles Δ varies.]

Let Δe_t represent the total system error due to quantization. Then,

$$|\Delta e_t| = |\Delta e_1 + \Delta e_2 + \ . \ . \ . + \Delta e_1| \leq |\Delta e_1| + |\Delta e_2| + \ . \ . \ . + |\Delta e_r|$$

$$\leq \hat{\rho} \sum_{j=1}^{r} || H_j ||, \text{ for all } k \geq k_o . \tag{3-51}$$

This corresponds to Johnson's (26).

III. DERIVATION OF TRANSFER FUNCTION $\frac{\phi_D}{\beta_c}(s)$ FROM SATURN V EQUATIONS OF MOTION

1. Introduction

The development of an open-loop transfer function from ϕ_D, the platform angle, to β_c, the input to the controller, is essential in the synthesis of a compensation function for the Saturn V control system. Therefore, in this chapter $\frac{\phi_D}{\beta_c}(s)$ is derived from the Saturn V, S-IC flight, equations of motion. The effects of sloshing in these equations are neglected in this preliminary effort to facilitate the use of signal flow techniques.

2. Derivation of $\frac{\phi_D}{\beta_c}(s)$

The transfer function $\frac{\phi_D}{\beta_c}(s)$ was derived from a signal flow representation of the simplified equations of motion. These equations, in generalized form, are:

1. Moment Equation

$$\ddot{\phi} = - C_1\alpha - C_2\beta - \frac{F l_{cg}}{I_{xx}} \sum_{i=1}^{4} Y'_i(X_\beta)\eta_i - \frac{F}{I_{xx}} \sum_{i=1}^{4} Y_i(X_\beta)\eta_i - \left(\frac{l_{cg} S_E}{I_{xx}} + \frac{I_E}{I_{xx}}\right)\ddot{\beta} - \frac{k_3 S_E}{I_{xx}}\beta \tag{4-1}$$

2. Normal Acceleration of Vehicle c.g.

$$\ddot{Z} = k_3\phi + k_4\beta + k_7\alpha + \sum_{i=1}^{4} \frac{F}{m} Y'(X_\beta)\eta_i + \frac{S_E}{m}\ddot{\beta} \tag{4-2}$$

3. Bending Mode Equations

$$\ddot{\eta}_i + 2\,\zeta_i\,\omega_i\,\dot{\eta}_i + \omega_i^2\eta_i = \frac{1}{m_i}\left\{R'Y\,(X_\beta)\beta + \left[S_E\,Y_i(X_\beta) - I_E Y'\,(X_\beta)\right]\ddot{\beta}\right. \tag{4-3}$$

$$i = 1,\ 2,\ 3,\ 4$$

4. Miscellaneous Equations

$$\phi_D = \phi + \sum_{i=1}^{4} Y'_i(X_D)\eta_i \tag{4-4}$$

$$\alpha = \phi - \frac{\dot{z}}{V} \tag{4-5}$$

$$\frac{\beta}{\beta_c} = \frac{A_0 + A_1 s}{B_0 + B_1 s + B_2 s^2 + B_3 s^3 + B_4 s^4 + B_5 s^5 + B_6 s^6 + B_7 s^7} \tag{4-6}$$

The signal flow graph representing the Saturn V simplified equations of motion is shown in Figure 4. From this signal flow graph, the transfer function $\frac{\phi_D}{\beta_c}(s)$ may be derived by the use of Mason's gain formula.

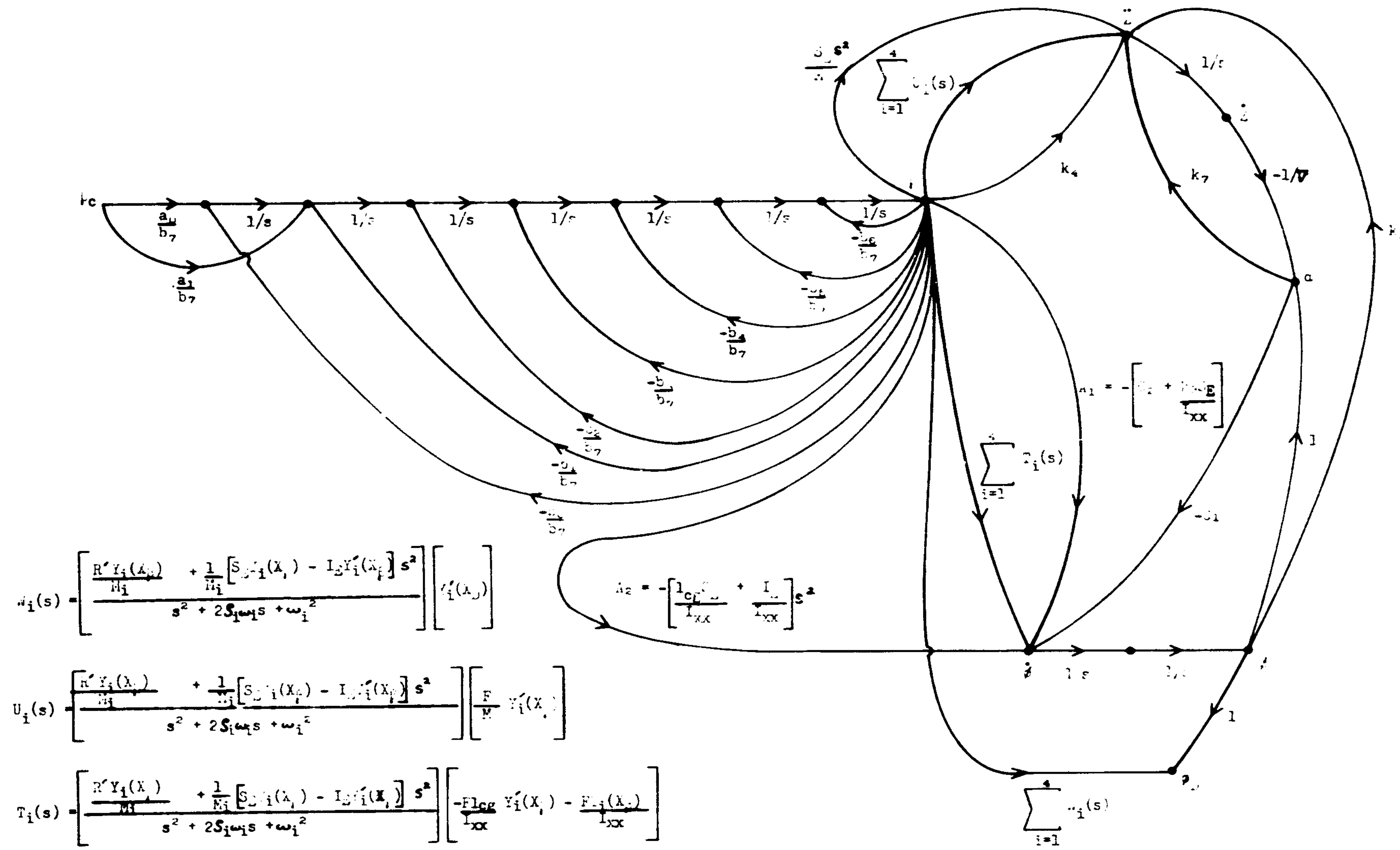

Fig. 4.- Signal Flow Graph Representation of Saturn V S-IC Equations of Motion with Slosh Neglected

The general form of $\frac{\Phi_D}{\beta_c}(s)$ from Mason's gain formula is

$$\frac{\Phi_D}{\beta_c}(s) = \frac{Z_4\ (Z_0 s^4 + \dots + Z_3 s + 1.0)}{D_{10}\ (D_0 s^{10} + \dots + D_9 s + 1.0)} + \sum_{i=1}^{4} \frac{Y_{6i}\ (Y_{0i} s^6 + \dots + Y_{5i} s + 1.0)}{D_{10}\ \omega_i^2\ (\frac{s^2}{\omega_i^2} + \frac{2\zeta_i s}{\omega_i} + 1.0)(D_0 s^{10} + \dots + D_9 s + 1.0)} \qquad (4\text{-}7)$$

Nyquist Program

A digital program has been written for use on an IBM 7040 digital computer as an aid in generating a system compensation function. The program, which appears in the Appendix, has three functions. They are

1. Calculation of $\frac{\Phi_D}{\beta_c}(s)$ transfer function coefficients.

2. Calculation of the poles $\frac{\Phi_D}{\beta_c}(s)$.

3. Calculation of the sampled frequency response of $H_0\ \frac{\Phi_D^*}{\beta_c}(s)$, where H_0 represents the transfer function of a zero-order hold.

The frequency response is computed by the series approximation method for positive frequencies only. Thus, the image plot must be generated external to the program to complete the Nyquist diagram for the open-loop system.

V. CONCLUSIONS

It was shown for the case where two poles of the continuous transfer function are almost coincident and for the case where three poles of the continuous transfer function are almost coincident that numerical inaccuracies may be expected if the z-transforms of these functions are to be used to generate frequency response data. Criteria were developed to facilitate the generation of valid data for these two cases by specification of the required number of significant digits which must be carried for accurate computations. These criteria were based on the pole configuration and coefficients of the continuous transfer function. Four methods were suggested for the aversion of computational inaccuracies due to nearly coincident poles.

Several important conclusions were advanced which describe the effect of the finite word length characteristic of digital devices on the performance of discrete systems. It was shown that the incorporation of quantizers into an otherwise stable linear discrete system does not destroy the bounded input-bounded output stability exhibited by linear systems. However, such a system can exhibit limit cycle behaviour and system errors resulting from quantization are likely to exist. Several techniques for the estimation of the range of system errors due to quantization were considered and the acceptability of the results obtained by their application was discussed. A detailed development of a method, which can be used for the computation of a dynamic upper bound on quantization errors, was given. The method is founded on Liapunov's Direct Method and is applicable to discrete systems which are linear if quantization is excluded and whose eigenvalues are less than unity in magnitude.

A transfer function from the control engine input signal to the attitude error angle was developed from the Saturn V equations of motion excluding vehicle slosh dynamics. A digital computer program was generated to facilitate development of the coefficients, the roots of the characteristic equation and the sampled frequency response for the above transfer function.

REFERENCES

1. G. W. Johnson, "Upper Bound on Dynamic Quantization Error in Digital Control Systems Via the Direct Method of Liapunov", IEEE Transactiona on Automatic Control, vol. AC-10, No. 4 pp. 439-448, Oct. 1965.

2. G. W. Johnson, G.N.T. Lack, "Comments on Upper Bound on Dynamic Quantization Error in Digital Control Systems Via the Direct Method of Liapunov", IEEE Transactions on Automatic Control, vol. AC-11, No. 2, pp. 331-334, April, 1966.

3. B. Widrow, "Statistical Analysis of Amplitude Quantized Sampled Data Systems", Transactions AIEE on Applications and Industry, No. 52, Jan. 1961, pp 555-568.

4. B. Widrow, "A Study of Rough Amplitude Quantization by Means of Nyquist Sampling Theory", IRE Transactions on Circuit Theory, vol. CT-3, Dec. 1964, pp. 266-276.

5. J. E. Bertram, "The Effect of Quantization in Sampled-Feedback Systems", Trans. AIEE on Applications and Industry, pt-2, vol. 77, pp. 177-181, Sept. 1958.

6. J. B. Slaughter, "Quantization Errors in Digital Control Systems", IEEE Transactions on Automatic Control, vol. AC-9, pp. 70-74, Jan. 1964.

7. G. W. Johnson, "On Liapunov Stability vs. Bounded Input-Bounded-Output", IEEE Transactions on Automatic Control, vol. AC-9, pp. 178-179, April, 1964.

8. W. Hahn, Theory and Application of Liapunov's Direct Method, Englewood Cliffs, N.J., Prentice-Hall, 1963.

9. R. S. Varga, Matrix Iterative Analysis, Englewood Cliffs, N.J., Prentice-Hall, 1962.

10. L. J. LaSalle and S. Lefschetz, Stability by Liapunov's Direct Method With Applications, New York, Academic Press, 1961.

11. L. Zadeh and C. Desoer, Linear System Theory, New York, McGraw-Hill, 1963.

APPENDIX

COMPUTER PROGRAM FOR CALCULATION OF Φ_D/β_C

```
**************************************************************************************

                Z4*(Z0*S**4 + Z1*S**3 + Z2*S**2 + Z3*S + 1.0)
PHID/BETAC=-------------------------------------------------------------------------/
                D1 *(D0*S**10 + D1*S**9 + D2*S**8 + D3*S**7 + D4*S**6 + D5*S**5

                /------------------------------------------------- + SUM(I=1 THRU 4)
                + D6*S**4 + D7*S**3 + D8*S**2 + D9*9 + 1.0)

                YI6*(YI0*S**6 + YI1*S**5 + YI2*S**4 + YI3*S**3 + YI4*S**2 +
                ----------------------------------------------------------------------/
                WI**2*D10*(S**2/WI**2 + S*2*ZETAI/WI + 1.0)*(D0*S**10 + D1*S**9

                YI5*S + 1.0)
                /---------------------------------------------------------------------/
                + D2*S**8 + D3*S**7 + D4*S**6 + D5*S**5 + D6*S**4 + D7*S**3 +


                /----------------------
                D8*S**2 + D9*S + 1.0)
**************************************************************************************

      DIMENSION YB(4), YPB(4), YPD(4), GM(4), W(4), ZETA(4),
     1AR(10), AI(10), SAVE(18), SSAVE(18), Q1(4), Q2(4), Q3(4), Q4(4),
     2P1(4), Y0(4), Y1(4), Y2(4), Y3(4), Y4(4), Y5(4), Y6(4),
     3P1R(4), P2R(4), P1I(4), P2I(4)
      COMPLEX TF,S,DEN1,NUM1,DD(4),NN(4),CMPLX,CEXP
   10 FORMAT(3(5X,E15.8))
   11 FORMAT(4(5X,E15.8))
      READ(5,10)AL0,AL1,B0
      READ(5,10)B1,B2,B3
      READ(5,10)B4,B5,B6
      READ(5,11)B7,C1,C2,F
      READ(5,11)ALCG, AIXX, AIF, T
      READ(5,11)ASE, AK3, AK4, P
      READ(5,11)AK7, AM, RP, V
      READ(5,11)W(1), W(2), W(3), W(4)
      READ(5,11)ZETA(1), ZETA(2), ZETA(3), ZETA(4)
      READ(5,11)GM(1), GM(2), GM(3), GM(4)
      READ(5,11)YB(1), YB(2), YB(3), YB(4)
      READ(5,11)YPB(1), YPB(2), YPB(3), YPB(4)
      READ(5,11)YPD(1), YPD(2), YPD(3), YPD(4)
   33 FORMAT(1H ,4X,10HINPUT DATA,//)
      PRINT 33
      PRINT 10, AL0, AL1, B0
      PRINT 10, B1, B2, B3
      PRINT 10, B4, B5, B6
      PRINT 11, B7, C1, C2, F
```

```
PRINT 11, ALCG, AIXX, AIE, T
PRINT 11, ASE, AK3, AK4, P
PRINT 11, AK7, AM, RP, V
PRINT 11, W(1), W(2), W(3), W(4)
PRINT 11, ZETA(1), ZETA(2), ZETA(3), ZETA(4)
PRINT 11, GM(1), GM(2), GM(3), GM(4)
PRINT 11, YB(1), YB(2), YB(3), YB(4)
PRINT 11, YPB(1), YPB(2), YPB(3), YPB(4)
PRINT 11, YPD(1), YPD(2), YPD(3), YPD(4)
AK3 = AK3*57.29578
AK4 = AK4*57.29578
AK7 = AK7*57.29578
A1 = -(C2+AK3*ASE/AIXX)
A2 = -(ALCG*ASE/AIXX+AIE/AIXX)
D10 = -B0*AK3*C1/(B7*V)
D0 = 1.0/D10
D1 = (B6/B7+AK7/V)/D10
D2 = (B5/B7+B6*AK7/(B7*V)+C1)/D10
D3 = (B4/B7+B6*C1/B7+B5*AK7/(B7*V)-AK3*C1/V)/D10
D4 = (B3/B7-B6*AK3*C1/(B7*V)+B5*C1/B7+B4*AK7/(B7*V))/D10
D5 = (B2/B7-B5*AK3*C1/(B7*V)+B4*C1/B7+B3*AK7/(B7*V))/D10
D6 = (B1/B7-B4*AK3*C1/(B7*V)+B3*C1/B7+B2*AK7/(B7*V))/D10
D7 = (B0/B7-B3*AK3*C1/(B7*V)+B2*C1/B7+B1*AK7/(B7*V))/D10
D8 = (-B2*AK3*C1/(B7*V)+B1*C1/B7+B0*AK7/(B7*V))/D10
D9 = (-B1*AK3*C1/(B7*V)+B0*C1/B7)/D10
Z4 = A1*AL0*AK7/(B7*V)+C1*AK4*AL0/(B7*V)
Z0 = (A2*AL1/B7)/Z4
Z1 = (A2*AL0/B7+A2*AL1*AK7/(B7*V)+ASE*C1*AL1/(B7*AM*V))/Z4
Z2 = (A2*AL0*AK7/(B7*V)+A1*AL1/B7+ASE*C1*AL0/(B7*AM*V))/Z4
Z3 = (A1*AL0/B7+A1*AL1*AK7/(B7*V)+C1*AK4*AL1/(B7*V))/Z4
DO 1 I=1, 4
W(I) = W(I)*2.0*3.1415927
Q1(I) = (ASE*YB(I)-AIE*YPB(I))/GM(I)
Q2(I) = P*E*YPB(I)/AM
Q3(I) = P*(-E*ALCG/AIXX*YPB(I)-E*YB(I)/AIXX)
Q4(I) = YPD(I)
P1(I) = RP*YB(I)/GM(I)
Y6(I) = Q1(I)*AL0/(B7*V)*(Q2(I)*C1+Q3(I)*AK7-Q4(I)*AK3*C1)
Y0(I) = (Q4(I)*P1(I)*AL1/B7)/Y6(I)
Y1(I) = (Q4(I)*P1(I)/B7*(AL0+AL1*AK7/V))/Y6(I)
Y2(I) = (AL1/B7*(Q3(I)*P1(I)+Q1(I)*Q4(I))+Q4(I)*P1(I)/B7*(AL0*AK7/
1         V+AL1*C1))/Y6(I)
Y3(I) = (P1(I)/B7*(Q2(I)*C1*AL1/V+Q3(I)*AL0+Q3(I)*AL1*AK7/V)+
1         Q4(I)/B7*(Q1(I)*AL0+Q1(I)*AL1*AK7/V+P1(I)*AL0*C1-P1(I)*
2         AL1*AK3*C1/V))/Y6(I)
Y4(I) = (Q2(I)*P1(I)*C1*AL0/V+Q1(I)*Q3(I)*AL1+Q3(I)*P1(I)*AL0*AK7/
1         V+Q1(I)*Q4(I)*AL0*AK7/V+Q1(I)*Q4(I)*AL1*C1-Q4(I)*P1(I)*AL0
2         *AK3*C1/V)/(B7*Y6(I))
```

```
    1 Y5(I) =  Q1(I)/Q7*(Q2(I)*C1*AL1/V+Q3(I)*AL0+Q2(I)*AL1*AK7/V+Q4(I)*
     1          AL0*C1-Q4(I)*AL1*AK3*C1/V)/Y6(I)
  111 FORMAT(1H1,4X,43HTHE COMPUTED COEFFICIENTS OF PHID/BETAC FOR,//)
      PRINT 111
   12 FORMAT(5X,12HFLIGHT TIME= F5.1,1X,11HSECONDS ARE,//)
      PRINT 12, T
   13 FORMAT(5X,3HZ0=F15.8,8X,3HZ1=F15.8,8X,3HZ2=F15.8,//)
      PRINT 13, Z0, Z1, Z2
   14 FORMAT(5X,3HZ3=F15.8,8X,3HZ4=F15.8,//)
      PRINT 14, Z3, Z4
   15 FORMAT(5X,3HD0=F15.8,8X,3HD1=F15.8,8X,3HD2=F15.8,//)
      PRINT 15, D0, D1, D2
   16 FORMAT(5X,3HD3=F15.8,8X,3HD4=F15.8,8X,3HD5=F15.8,//)
      PRINT 16, D3, D4, D5
   17 FORMAT(5X,3HD6=F15.8,8X,3HD7=F15.8,8X,3HD8=F15.8,//)
      PRINT 17, D6, D7, D8
   18 FORMAT(5X,3HD9=F15.8,8X,4HD10=F15.8,//)
      PRINT 18, D9, D10
   19 FORMAT(5X,6HY0(1)=F15.8,5X,6HY1(1)=F15.8,5X,6HY2(1)=F15.8,5X,
     16HY3(1)=F15.8,//)
  119 FORMAT(5X,6HY4(1)=F15.8,5X,6HY5(1)=F15.8,5X,6HY6(1)=F15.8,//)
      PRINT 19, Y0(1), Y1(1), Y2(1), Y3(1)
      PRINT 119, Y4(1), Y5(1), Y6(1)
   20 FORMAT(5X,6HY0(2)=F15.8,5X,6HY1(2)=F15.8,5X,6HY2(2)=F15.8,5X,
     16HY3(2)=F15.8,//)
  120 FORMAT(5X,6HY4(2)=F15.8,5X,6HY5(2)=F15.8,5X,6HY6(2)=F15.8,//)
      PRINT 20, Y0(2), Y1(2), Y2(2), Y3(2)
      PRINT 120, Y4(4), Y5(2), Y6(2)
   21 FORMAT(5X,6HY0(3)=F15.8,5X,6HY1(3)=F15.8,5X,6HY2(3)=F15.8,5X,
     16HY3(3)=F15.8,//)
  121 FORMAT(5X,6HY4(3)=F15.8,5X,6HY5(3)=F15.8,5X,6HY6(3)=F15.8,//)
      PRINT 21, Y0(3), Y1(3), Y2(3), Y3(3)
      PRINT 121, Y4(3), Y5(3), Y6(3)
   22 FORMAT(5X,6HY0(4)=F15.8,5X,6HY1(4)=F15.8,5X,6HY2(4)=F15.8,5X,
     16HY3(4)=F15.8,//)
  122 FORMAT(5X,6HY4(4)=F15.8,5X,6HY5(4)=F15.8,5X,6HY6(4)=F15.8,//)
      PRINT 22, Y0(4), Y1(4), Y2(4), Y3(4)
      PRINT 122, Y4(4), Y5(4), Y6(4)
C     N IS THE ORDER OF THE POLYNOMIAL
 1112 N = 9
      NZ = N
      IM=0.0
      C = 1.0
      SF = 1.0
      AR(11) = 1.0*C
      AR(10) = D9/SF*C
      AR(9) = D8/SF**2*C
      AR(8) = D7/SF**3*C
      AR(7) = D6/SF**4*C
```

```
      AR(6) = D5/SF**5*C
      AR(5) = D4/SF**6*C
      AR(4) = D3/SF**7*C
      AR(3) = D2/SF**8*C
      AR(2) = D1/SF**9*C
      AR(1) = D0/SF**10*C
      CALL SOLVER(N,IM,AR,AI,SAVE)
  222 FORMAT(1H1,4X,40HTHE POLES OF PHID/BETAC (RIGID BODY) ARE ,//)
      PRINT 222
      DO 3 K = 1, NZ
      L=2*K-1
      M=2*K
      SSAVE(L) = SAVE(L) / SF
      SSAVE(M) = SAVE(M) / SF
      IF(ABS(SSAVE(M))-0.10000000E-08) 34,34,35
   34 SSAVE(M) = 0.00000000E 00
   35 CONTINUE
    2 FORMAT(5X,E15.8,2X,E15.8,2H J,//)
      WRITE(6,2) SSAVE(L), SSAVE(M)
    3 CONTINUE
   76 FORMAT(1H1,4X,46HTHE POLES RESULTING FROM THE BENDING MODES ARE
     1,//)
      PRINT 76
      DO 81 I=1,4
      P1R(I) = -ZETA(I)*W(I)
      P1I(I) = -W(I)*SQRT(1.0-ZETA(I)**2)
      P2R(I) = P1R(I)
      P2I(I) = -P1I(I)
      WRITE(6,2) P1R(I), P1I(I)
      WRITE(6,2) P2R(I), P2I(I)
   81 CONTINUE
   66 FORMAT(1H1,4X,30HFREQUENCY RESPONSE OUTPUT DATA ,//)
      PRINT 66
      OMEG = 0.005
    4 TF = CMPLX(0.0, 0.0)
      TS = 0.04
      OMEGS = 2.0*3.1415927/TS
      NTILT = 2
      NX = 2*NTILT+1
      DO 24 J = 1, NX
      XJ = -NTILT+J-1
      OMEG1 = OMEG+XJ*OMEGS
      S = CMPLX(0.0, OMEG1)
      NUM1 = (Z0+Z1/S+Z2/S**2+Z3/S**3+1.0/S**4)*0.1E-10
      DEN1 = D10/Z4*(D0*S**6+D1*S**5+D2*S**4+D3*S**3+D4*S**2+D5*S
     1          +D6PD7/S+D8/S**2+D9/S**3+1.0/S**4)*0.1E-10
      TF = TF-1.0/TS*NUM1/DEN1*(1.0-CEXP(-S*TS))/S
      DO 23 K=1,4
      NN(K) = Y6(K)*(Y0(K)*S+Y1(K)+Y2(K)/S+Y3(K)/S**2+Y4(K)/S**3
     1          +Y5(K)/S**4+1.0/S**5)*.1E-20
```

```
      DD(K) = W(K)**2*(S**2/W(K)**2+ 2.0*ZETA(K)/W(K)+1.0)*DEN1/S*24
     1          *.1E-20
   23 TF = TF +1.0/TS*NN(K)/DD(K)*(1. -CEXP(-S*TS))/S
   24 CONTINUE
      ABSVAL = CABS(TF)
      DB = 20.0*ALOG10 (ABSVAL)
      PHASE = 57.29578*ATAN2(AIMAG(TF), REAL(TF))
      IF(PHASE) 30,30,31
   30 PHASE = PHASE+360.0
   31 CONTINUE
   25 FORMAT(5X,6HOMEGA=F7.3,5X,3HDB=F9.3,5X,6HPHASE=F9.3)
      PRINT 25,OMEG,DB,PHASE
      EXIT = 3.1415927/TS
      IF(OMEG .GE. 0.0 .AND. OMEG .LE. 0.2) GO TO 44
      OMEG = OMEG + 0.2
      GO TO 55
   44 OMEG = OMEG + .01
   55 IF(OMEG-EXIT) 4,4,7
    7 CONTINUE
      STOP
      END
$IBFTC SOLVER
      SUBROUTINE SOLVER(N,IM,AR,AI,SAVE)
      DIMENSION AR(10), AI(10), SAVE(18)
C     BEGIN DATA INPUT
      M=N+1
      IF (IM)24,23,24
 23   DO 30 J=1,M
      AI(J)=0.0
30    CONTINUE
 24   INDEX=2
C     END DATA INPUT
C     BEGIN MULLERS METHOD
5     KOUNT =0
      TOL = 1.E-20
      ITR8=50
      P1R=AR(N+1)+AR(N)+AR(N-1)
      P1I=AI(N+1)+AI(N)+AI(N-1)
      P2R=AR(N+1)
      P2I=AI(N+1)
      U=AR(N)**2-AI(N)**2-4.0*(AR(N-1)*AR(N+1)-AI(N-1)*AI(N+1))
      V=2.0*AR(N)*AI(N)-4.0*(AR(N-1)*AI(N+1)+AI(N-1)*AR(N+1))
      Q=AR(N)
      R=AI(N)
      KRAD=1
      GO TO 500
50    TEMP=RADR**2+RADI**2
      IF(TEMP) 510,510,511
510   XR=.5
      XI=0.0
```

```
      GO TO 512
511   XR=-2.0*(AR(N+1)*RADR+AI(N+1)*RADI)/TEMP
      XI=-2.0*(AI(N+1)*RADR-AR(N+1)*RADI)/TEMP
512   HR=XR
      HI=XI
      WR=-XR
      WI=-XI
51    KPOLY=1
      GO TO 41
504   TEMP=(ABS(PR)+ABS(PI))/(ABS(P2R)+ABS(P2I))
      IF(TEMP-10.)505,505,506
506   WR=.5*WR
      WI=.5*WI
      HR=-WR
      HI=-WI
      XR=HR
      XI=HI
      GO TO 51
505   P3R=PR
      P3I=PI
52    DEN=ABS(P3R)+ABS(P3I)
      IF(DEN)7,7,5211
5211  XRO=XR
      XIO=XI
      IF(ABS(P1R-P2R)-1.E-25)521,521,525
521   IF(ABS(P1I-P2I)-1.E-25)522,522,525
522   IF(ABS(P1R-P3R)-1.E-25)523,523,525
523   IF(ABS(P1I-P3I)-1.E-25)524,524,525
524   WR=1.0
      WI=0.0
      GO TO 53
525   TEMP=WR+1.0
      DR=TEMP*P3R-WI*P3I
      DI=TEMP*P3I+WI*P3R
      Q=TEMP*P2R-WI*P2I
      R=TEMP*P2I+WI*P2R
      TEMP=WR*P1R-WI*P1I-Q+P3R
      FI=WR*P1I+WI*P1R-R+P3I
      FR=WR*TEMP-WI*FI
      FI=WR*FI+WI*TEMP
      Q=FR-Q+DR
      R=FI-R+DI
      U=Q*Q-R*R-4.0*(DR*FR-DI*FI)
      V=2.0*Q*R-4.0*(DR*FI+DI*FR)
      KRAD=2
      GO TO 500
526   TEMP=RADR**2+RADI**2
      WR=-2.0*(DR*RADR+DI*RADI)/TEMP
      WI=-2.0*(DI*RADR-DR*RADI)/TEMP
53    HRO=HR
      HIO=HI
```

```
      HR=WR*HRO-WI*HIO
      HI=WR*HIO+WI*HRO
      XR=HR+XRO
      XI=HI+XIO
      KPOLY=2
      GO TO 41
 527  TEMP=(ABS(PR)+ABS(PI))/DEN
      IF(TEMP-10.)6,6,528
528   WR=0.5*WR
      WI=0.5*WI
      HR=HRO
      HI=HIO
      GO TO 53
6     KOUNT=KOUNT+1
C     APPLY CONVERGENCE CRITERION
      TEST=ABS(XR-XRO)+ABS(XI-XIO)
      TEMP=ABS(XR)+ABS(XI)
      IF(TEMP-1.0)62,62,61
61    TEST=TEST/TEMP
62    IF(TEST-TOL)7,7,64
64    P1R=P2R
      P1I=P2I
      P2R=P3R
      P2I=P3I
      P3R=PR
      P3I=PI
      IF(KOUNT-ITR8+4)52,541,541
 541  IF(KOUNT-ITR8)52,543,543
C     SET 552 FOR NEW ITR8 AND NEW TOL IF DESIRED.
543   ITR8=ITR8+10
      TOL=TOL*10.
      GO TO 52
 7    SAVE(INDEX-1)=XR
      SAVE(INDEX)=XI
      INDEX=INDEX+2
C     SUBROUTINE FOR CALC. OF COEFS. OF POLY. OF DEGREE N-1 FORM THOSE OF
C     POLY. OF DEGREE N WHEN DIVIDED BY FACTOR X-(XR+I*XI). ALL NOS. ARE
C     COMPLEX. ARGUMENTS ARE N,XR,XI,AR(J),AI(J),J=1,...,N+1.
C     RESULTS OF CALC. ARE AR(J),AI(J),J=1,...,N,AND REMR,REMI.
      M=N+1
      DO 76 J=2,M
      AR(J)=AR(J)+XR*AR(J-1)-XI*AI(J-1)
76    AI(J)=AI(J)+XR*AI(J-1)+XI*AR(J-1)
C     END SUBROUTINE
      N=N-1
      IF(N-1)9,81,5
81    TEMP=AR(1)**2+AI(1)**2
      XR=-(AR(2)*AR(1)+AI(2)*AI(1))/TEMP
      XI=(AR(2)*AI(1)-AI(2)*AR(1))/TEMP
      SAVE(INDEX-1)=XR
```

```
      SAVE(INDEX)=XI
    9 RETURN
C      SUBROUTINE FOR THE CALC.
C      (RADR,RADI)=(Q,R)+SQR(U,V) OF GREATEST MAGNITUDE.
  500 TEMP=SQRT(U*U+V*V)
      IF(TEMP+U)5001,5001,5002
5001  RADR=0.0
      GO TO 5003
 5002 RADR=SQRT(TEMP+U)
5003  IF(TEMP-U)5004,5004,5005
5004  RADI=0.0
      GO TO 5006
 5005 RADI=SQRT(TEMP-U)
5006  TEMP=Q*RADR+R*RADI
      IF(TEMP)501,501,502
501   RADR=Q-RADR/1.4142135
      RADI=R-RADI/1.4142135
      GO TO 503
502   RADR=Q+RADR/1.4142135
      RADI=R+RADI/1.4142135
503   GO TO (50,526),KRAD
C     END SUBROUTINE
C     SUBROUTINE FOR EVALUATION OF P(X),X COMPLEX, THAT IS,
C     (PR,PI)=(...((A(1)*X+A(2))*X+A(3))...)*X+A(N+1)
C     ARGUMENTS ARE N,XR,XI,AR(J),AI(J).
41    M=N+1
      PR=AR(1)
      PI=AI(1)
      DO 42 J=2,M
      TEMP=XI*PR+XR*PI+AI(J)
      PR=XR*PR-XI*PI+AR(J)
      PI=TEMP
      IF(ABS(PR)+ABS(PI)-1.E37)42,42,420
42    CONTINUE
 420  GO TO(504,527),KPOLY
C     END SUBROUTINE
      END
$ENTRY
```